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The transient motion of a floating body

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An analytical method of calculating the body motion was given in an earlier paper. Viscosity and surface tension were neglected, and the equations of motion were linearized. It was found that, for a half-immersed horizontal circular cylinder of radius a, the vertical motion at time $\tau(a/g)^{\frac{1}{2}}$ is described by the functions $h_1(\tau)$ (for an initial velocity) and $h_2(\tau)$ (for an initial displacement) where

$$h_1(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-iu\tau}du}{1-\frac{1}{4}\pi u^2(1+\Lambda(u))}$$

and

$$h_2(\tau) = -\tfrac{1}{8}i\int_{-\infty}^{\infty} \frac{u(1+\Lambda(u))\,e^{-iu\tau}du}{1-\frac{1}{4}\pi u^2(1+\Lambda(u))}.$$

The function $\Lambda(u)$ in these integrals is the force coefficient which describes the action of the fluid on the body in a forced periodic motion of angular frequency $u(g/a)^{\frac{1}{2}}$. To determine $\Lambda(u)$ for any one value of u an infinite system of linear equations must be solved.

In the present paper a numerical study is made of the functions $h_1(\tau)$ and $h_2(\tau)$. The integrals defining $h_1(\tau)$ and $h_2(\tau)$ are not immediately suitable for numerical integration, for small τ because the integrands decrease slowly as u increases, for large τ because of the oscillatory factor $e^{-iu\tau}$. It is shown how these difficulties can be overcome by using the properties of $\Lambda(u)$ in the complex u-plane. It is found that after an initial stage the motion of the body is closely approximated by a damped harmonic oscillatory motion, except during a final stage of decay when the motion is non-oscillatory and the amplitude is very small. It is noteworthy that the motion of the body can be found accurately, although little can be said about the wave motion in the fluid.

1. Introduction

Consider a rigid body floating on the free surface of a fluid, which is slightly disturbed from its position of stable equilibrium. The ensuing free motion consists of a motion of the body, together with a wave motion of the fluid which (if the fluid is unbounded horizontally) progressively carries energy away from the body to infinity. Ultimately the body and the fluid return to their equilibrium state of rest. In a previous paper (Ursell 1964, hereafter referred to as I) the transient motion of the body was studied analytically. Viscosity and surface tension were neglected, and the equations of motion were linearized. The method described in I is applicable to bodies of arbitrary shape in two or three dimensions, and was applied in I to the heaving (i.e. vertical) motion of a half-immersed circular cylinder of radius a. The free motion of the cylinder was regarded as the superposition of simple harmonic motions, and the vertical displacement of the body was thus obtained in the form of Fourier integrals. These results are summarized in §3 below where two cases are considered. The first case (initial velocity) is described by the function $h_1(\tau)$, the second case (initial displacement) by the function $h_2(\tau)$. Both these functions involve the complex-valued *force coefficient* $\Lambda(u)$ which describes the hydrodynamic force exerted by the fluid on the body in a forced periodic motion of real angular frequency $u(g/a)^{\frac{1}{2}}$ and of constant amplitude, see §2 below. The present paper will be devoted to a numerical study of the functions $h_1(\tau)$ and $h_2(\tau)$.

2. Equations of motion

The statements in this paragraph are taken from I, where a detailed derivation is given. It is assumed that the equilibrium position of the centre of the circular cylinder is in the mean free surface. This point is taken as the origin of rectangular Cartesian co-ordinates. Polar co-ordinates are defined by $x = r\sin\theta$, $y = r\cos\theta$, where the ray $\theta = 0$ is taken along the downward vertical. The equilibrium position of the cylinder is $r = a$ where a is the radius of the cylinder. The vertical displacement $y_0(t)$ of the cylinder is to be found. The amplitude of motion is assumed to be so small that all equations can be linearized. Since viscosity is neglected and the density ρ of the fluid is constant it is possible to describe the motion of the fluid by a velocity potential $\phi(x, y; t)$ which by symmetry must be an even function of θ and which satisfies the equation of continuity.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi(x, y; t) = 0 \quad \text{when} \quad r > a \text{ and } y > 0. \tag{2.1}$$

The linearized condition of constant pressure at the free surface is

$$\frac{\partial^2\phi}{\partial t^2}-g\frac{\partial\phi}{\partial y} = 0 \quad \text{when} \quad y = 0,\ |x| > a \tag{2.2}$$

(cf. Lamb 1932, §227). On the cylinder the radial velocity components of the body and the fluid are equal,

$$\partial\phi/\partial r = \dot{y}_0(t)\cos\theta \quad \text{when} \quad r = a,\ -\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi. \tag{2.3}$$

Finally there is the equation of motion of the body

$$\tfrac{1}{2}\pi\rho a^2\ddot{y}_0(t) = -2\rho g a y_0(t)+\rho a\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\cos\theta\,[\partial\phi(a\sin\theta, a\cos\theta; t)/\partial t]\,d\theta+f_0(t), \tag{2.4}$$

where on the right-hand side the first term is the hydrostatic restoring force, the second is the resultant of the hydrodynamic pressures, and the third term is the applied vertical force. The mass of the body is $\frac{1}{2}\pi\rho a^2$ (per unit width), by the principle of Archimedes.

These were the equations which were solved in I by resolving the motion into its frequency components (see I, §3). The transform potential $\Phi(x,y;\omega)$ at angular frequency ω satisfies the equations (for the first problem)

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi(x,y;\omega) = 0 \quad \text{when} \quad r > a,\ y > 0; \tag{2.5}$$

$$(\omega^2+g\,\partial/\partial y)\Phi(x,y;\omega) = 0 \quad \text{when} \quad y = 0,\ |x| > a; \tag{2.6}$$

$$\partial\Phi/\partial r = -i\omega Y_0(\omega)\cos\theta \quad \text{when} \quad r = a,\ -\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi. \tag{2.7}$$

There is also the radiation condition

$$\frac{\partial\Phi}{\partial r}\mp\frac{i\omega^2}{g}\Phi\to 0 \quad \text{as} \quad r\to\infty \tag{2.8}$$

according as $\omega \gtrless 0$. Except for a normalizing constant the equations (2.5)–(2.8) are identical, for real ω, with the equations describing the fluid motion due to the forced periodic heaving of the circular cylinder with time factor $e^{-i\omega t}$ and constant amplitude $Y_0(\omega)$, see Ursell (1949). It is known that they define $\Phi(x,y;\omega)$ uniquely; evidently Φ is proportional to $Y_0(\omega)$. The non-dimensional *force coefficient* $\Lambda(\omega(a/g)^{\frac{1}{2}})$ is now defined by the equation

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\Phi(a\sin\theta,\ a\cos\theta;\ \omega)\cos\theta\, d\theta = \tfrac{1}{2}\pi i a\omega Y_0(\omega)\Lambda(\omega(a/g)^{\frac{1}{2}}). \tag{2.9}$$

The function Λ is thus in principle a known function which, for real ω, can be deduced from published computations on periodic heaving (e.g. Ursell 1957). In fact the real part of $\Lambda(\omega(a/g)^{\frac{1}{2}})$ is the usual virtual-mass coefficient, the imaginary part of $\Lambda(\omega(a/g)^{\frac{1}{2}})$ is

$$\pm\frac{2}{\pi}\left(\frac{g}{a\omega^2}\right)^2\left(\frac{A(\omega)}{Y_0(\omega)}\right)^2 \text{according as } \omega\gtrless 0,$$

where $A(\omega)$ is the wave amplitude at infinity, and $A(\omega)/Y_0(\omega)$ is the usual wave-making coefficient. For further properties of Φ and Λ see I, pp. 314–318.

3. Analytical results and properties of $\Lambda(u)$, $h_1(\tau)$ and $h_2(\tau)$

Two problems were treated in I.

First problem: the cylinder is set in motion by a downward force $f_0(t)$ per unit width. The downward displacement $y_1(t)$ of the cylinder at time t was found to be (I, equation (3.10))

$$y_1(t) = \frac{1}{2\rho g^{\frac{1}{2}}a^{\frac{3}{2}}}\int_0^t f_0(t')\,h_1((t-t')\,(g/a)^{\frac{1}{2}})\,dt', \tag{3.1}$$

where
$$h_1(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-iu\tau}\,du}{1-\frac{1}{4}\pi u^2(1+\Lambda(u))}. \tag{3.2}$$

In particular, if the motion is started impulsively at time $t=0$, then it is evident that the subsequent displacement $\mathscr{y}_1(t)$ is given by

$$\mathscr{y}_1(t) = \frac{\mathscr{T}}{2\rho g^{\frac{1}{2}}a^{\frac{3}{2}}}h_1(t(g/a)^{\frac{1}{2}}), \tag{3.3}$$

where the total impulse per unit width is

$$\mathcal{T} = \int_0^\infty f_0(t')\,dt',$$

and the integration in fact extends over a short time. Equation (3.3) gives an immediate physical meaning to $h_1(\tau)$. In terms of the initial velocity $\dot{y}_1(0)$ the displacement $y_1(t)$ takes the form

$$y_1(t) = \tfrac{1}{2}\pi(a/g)^{\frac{1}{2}}\dot{y}_1(0)\,h_1(t(g/a)^{\frac{1}{2}}) \tag{3.4}$$

since $h_1'(\tau)\to 2/\pi$ as $\tau\to 0$; see (3.10) below.

Second problem: the cylinder is released from rest with an initial displacement $y_2(0)$

The displacement $y_2(t)$ at time t was found to be (I, equation (4.9))

$$y_2(t) = y_2(0)\,h_2(t(g/a)^{\frac{1}{2}}),$$

where

$$h_2(\tau) = -\tfrac{1}{8}i\int_{-\infty}^{\infty}\frac{u(1+\Lambda(u))\,e^{-iu\tau}}{1-\frac{1}{4}\pi u^2(1+\Lambda(u))}\,du. \tag{3.5}$$

It was shown that $h_1(\tau) = -h_2'(\tau)$. In the integrals defining $h_1(\tau)$ and $h_2(\tau)$ the contour of integration is the real u-axis indented to pass above $u=0$. The following properties were also obtained in I. The function $\Lambda(u)$ is defined in the first place for real values of u but the definition of $\Lambda(u)$ can be extended to the whole complex u-plane cut along the negative imaginary u-axis where $\Lambda(u)$ is single-valued but may possibly have poles. $\Lambda(u)$ is real along the positive imaginary u-axis and has a logarithmic infinity at $u=0$. The behaviour of $\Lambda(u)$ for large u in the upper half-plane was given by Ursell (1953), the behaviour in the lower half-plane is more difficult to obtain (see I, appendix 2). Crapper (1968) has since proved analytically that the asymptotic relation

$$\Lambda(u) \sim 1-(4/3\pi u^2)+\dots \quad\text{as}\quad u\to\infty, \tag{3.6}$$

which holds in the upper half-plane, continues to hold when $-\frac{1}{8}\pi < \arg u \leqslant 0$ and when $\pi \leqslant \arg u < \frac{9}{8}\pi$, and possibly in a larger angle. (Our numerical work shows that (3.6) is in fact valid in the larger angles $-\frac{1}{4}\pi < \arg u \leqslant 0$ and $\pi \leqslant \arg u < \frac{5}{4}\pi$.) Further properties of $\Lambda(u)$ are quoted in I, appendix.

We can infer the behaviour of $h_1(\tau)$ and $h_2(\tau)$ for large and small τ from the behaviour of the integrands for small and large u respectively. It was found in I that

$$h_1(\tau) \sim -8/\pi\tau^3 \quad\text{and}\quad h_2(\tau) \sim -4/\pi\tau^2 \quad\text{as}\quad \tau\to\infty. \tag{3.7}$$

For large u it is easy to see by using (3.6) that

$$\frac{1}{1-\frac{1}{4}\pi u^2(1+\Lambda(u))} \sim -\frac{2}{\pi u^2}-\frac{16}{3\pi^2u^4}, \tag{3.8}$$

and

$$\frac{u(1+\Lambda(u))}{1-\frac{1}{4}\pi u^2(1+\Lambda(u))} = -\frac{4}{\pi u}+\frac{4}{\pi u(1-\frac{1}{4}\pi u^2(1+\Lambda))}$$

$$\sim -\frac{4}{\pi u}-\frac{8}{\pi^2u^3}-\frac{64}{3\pi^3u^5}, \tag{3.9}$$

where these estimates are uniform with respect to arg u in the upper half u-plane and also in parts of the lower half-plane, as we have just noted. Also the functions on the left of (3.8) and (3.9) are regular in the whole of the upper half u-plane (I, p. 308). The asymptotic behaviour of $h_1(\tau)$ and $h_2(\tau)$ for small τ can therefore be found by term-by-term integration (Doetsch 1950, p. 503) where the path is indented to pass above $u = 0$. Thus

$$\begin{aligned} h_1(\tau) &\sim \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iu\tau}\left(-\frac{2}{\pi u^2}-\frac{16}{3\pi^2 u^4}\right)du \\ &= \frac{2}{\pi}\tau - \frac{8}{9\pi^2}\tau^3; \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} h_2(\tau) &\sim -\tfrac{1}{8}i\int_{-\infty}^{\infty} e^{-iu\tau}\left(-\frac{4}{\pi u}-\frac{8}{\pi^2 u^3}-\frac{64}{3\pi^3 u^5}\right)du \\ &= 1-\frac{1}{\pi}\tau^2+\frac{2}{9\pi^2}\tau^4. \end{aligned} \tag{3.11}$$

Higher terms in (3.10) and (3.11) can be obtained if higher terms in the asymptotic expansion (3.6) of $\Lambda(u)$ can be obtained. The next terms in (3.10) and (3.11) are believed to be of order τ^5 and τ^6 respectively.

4. Deformation of the contour of integration

The integrals (3.2) and (3.5) defining the functions $h_1(\tau)$ and $h_2(\tau)$ are not convenient for computation when the integration is taken along the real u-axis. (The following discussion will apply equally to $h_1(\tau)$ and $h_2(\tau)$, and will be confined to the right-hand half $\operatorname{Re} u > 0$ of the u-plane; the values of the integrands in the left-hand half are the complex conjugates of the values in the right-hand half. Additional notes on the computation are given in the appendix to the present paper.) On using the asymptotic estimate (3.6) for $\Lambda(u)$ it is seen that for large $|u|$ the integrands of $h_1(\tau)$ and $h_2(\tau)$ are of order $u^{-2}e^{-iu\tau}$ and $u^{-1}e^{-iu\tau}$ respectively, so that their magnitudes decrease very slowly along the real u-axis where the non-decreasing factor $e^{-iu\tau}$ oscillates rapidly for large τ. Also, the denominator $1-\frac{1}{4}\pi u^2(1+\Lambda)$ has a zero just below the positive real u-axis, at $u = u_0$, say. We have already noted (in §3 above) that $\Lambda(u)$, and therefore also the integrands, can be continued into the whole u-plane cut along the negative imaginary axis.

Accordingly the contour of integration from 0 to ∞ is now deformed into a contour C_+ which goes from 0 to ∞, which lies entirely in the fourth quadrant, which passes below the pole $u = u_0$, and which also satisfies the condition that no other pole lies between the real axis and C_+. The direction of the contour C_+ at ∞ is chosen close enough to arg $u = 0$ for the asymptotic relation

$$\Lambda(u) \sim 1-(4/3\pi u^2)+\dots \quad \text{as} \quad |u| \to \infty \tag{4.1}$$

to be valid between the real axis and C_+. It will be seen that there are such contours C_+. The gap at ∞ between C_+ and the real u-axis is closed at ∞ by a large circular arc.

Then, by Cauchy's theorem,

$$\int_0^\infty = \int_{C_+} - 2\pi i \text{ (residue at } u = u_0), \tag{4.2}$$

for the contribution from the large circular arc vanishes by Jordan's lemma (Titchmarsh 1939, §3.122). We must now specify C_+ more precisely. The greater the angle between C_+ and the real u-axis, the more rapidly does the exponential factor $e^{-iu\tau}$ in the integrand decrease along C_+. We have noted (see §3 above) that (4.1) is certainly valid when $-\frac{1}{8}\pi < \arg u \leqslant 0$, and that our computations show it to be valid in the larger angle $-\frac{1}{4}\pi < \arg u \leqslant 0$. We decided to take as

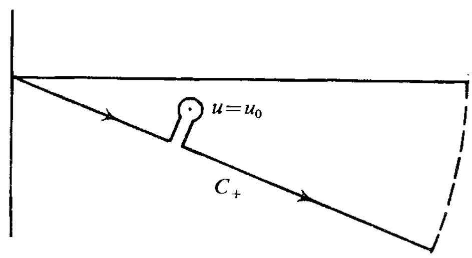

FIGURE 1. Deformation of the contour of integration.

our contour of integration C_+ the ray $\arg u = -\tan^{-1}\frac{1}{2}$; thus (4.1) is satisfied. It was verified, by computing $\Lambda(u)$ along a number of rays $\arg u = \text{const.}$, that $\Lambda(u)$ (which might have a pole) is regular between $\arg u = 0$ and $\arg u = -\tan^{-1}\frac{1}{2}$, and then it was shown, by using computed values of the function $\arg(1 - \frac{1}{4}\pi u^2(1+\Lambda))$ along $\arg u = 0$ and along $\arg u = -\tan^{-1}\frac{1}{2}$, that there is exactly one pole inside this angle. This was later confirmed by another check. For small τ the values of $h_1(\tau)$ and $h_2(\tau)$ were evaluated from (4.2) and also from the polynomials (3.10) and (3.11). If there were more than one pole in the angle then these values would differ by the residues from the additional poles. In fact they agreed very closely near $\tau = 0$.

The first term on the right-hand side of (4.2) will be called the *integral component*, the second term will be called the *polar component*. The polar component contains the factor $e^{-iu_0\tau}$ and thus represents a damped harmonic motion, and it will be seen that this is the dominant term during the greater part of the motion. Since it involves only the location of the pole $u = u_0$ and the values of the residues at u_0 it can be computed accurately by means of computations in the neighbourhood of u_0.

5. Results of the computations

We have prepared tables of $h_1(\tau)$ and $h_2(\tau)$ in the range $0 \leqslant \tau \leqslant 20$ which we believe to be correct to at least 3 places of decimals but which are too long to be included here. The functions $h_1(\tau)$ and $h_2(\tau)$ are shown graphically in figures 2 and 3 on which the polar components are also shown. It is seen that the functions are closely approximated by their polar (damped harmonic) components except during the first $1\frac{1}{2}$ cycles.

Figures 2 and 3 give only a rough idea, and greater accuracy may be obtained from figure 4 which shows the integral components $\hat{h}_1(\tau)$ and $\hat{h}_2(\tau)$ on a larger scale in the range $0 \leqslant \tau \leqslant 7$; this figure can be extended to $\tau = \infty$ by means of the relations $\hat{h}_1(\tau) \sim -(8/\pi)\tau^{-3}$ and $\hat{h}_2(\tau) \sim -(4/\pi)\tau^{-2}$ which are accurate to at least 2 places of decimals when $\tau > 7$. The values of $h_1(\tau)$ and $h_2(\tau)$ can then be found by

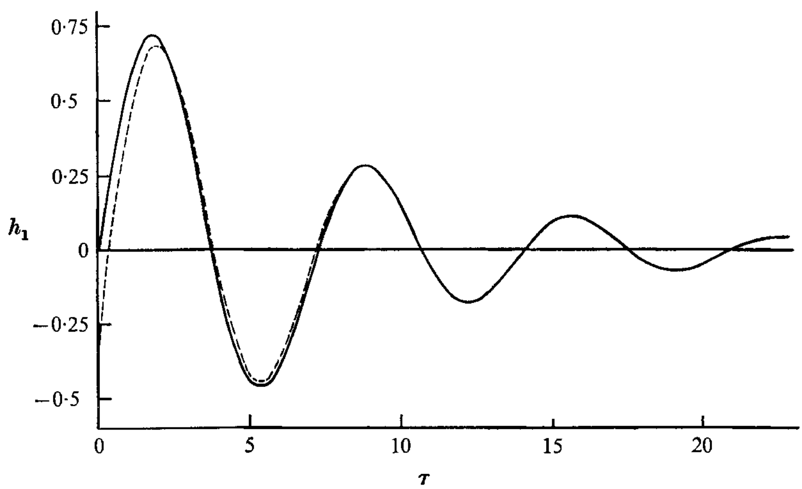

FIGURE 2. The function $h_1(\tau)$, case of initial velocity; - - - -, the polar component.

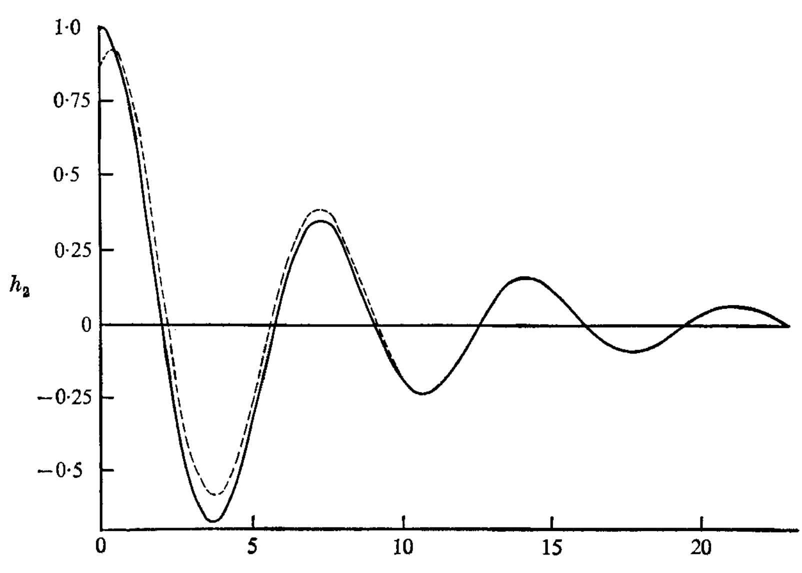

FIGURE 3. The function $h_2(\tau)$, case of initial displacement; - - - -, the polar component.

computing and adding the polar components which are given by the following expressions:

$$\text{polar component of } h_1(\tau) = 0{\cdot}8818 \exp(-0{\cdot}1309\tau) \sin(0{\cdot}9117\tau - 0{\cdot}3628), \quad (5.1)$$

$$\text{polar component of } h_2(\tau) = 0{\cdot}9664 \exp(-0{\cdot}1309\tau) \cos(0{\cdot}9117\tau - 0{\cdot}4805). \quad (5.2)$$

For large τ the polar components are exponentially small while the integral components are algebraically small. Thus ultimately the non-oscillatory integral components are dominant, as was noted in I. In particular the integral component

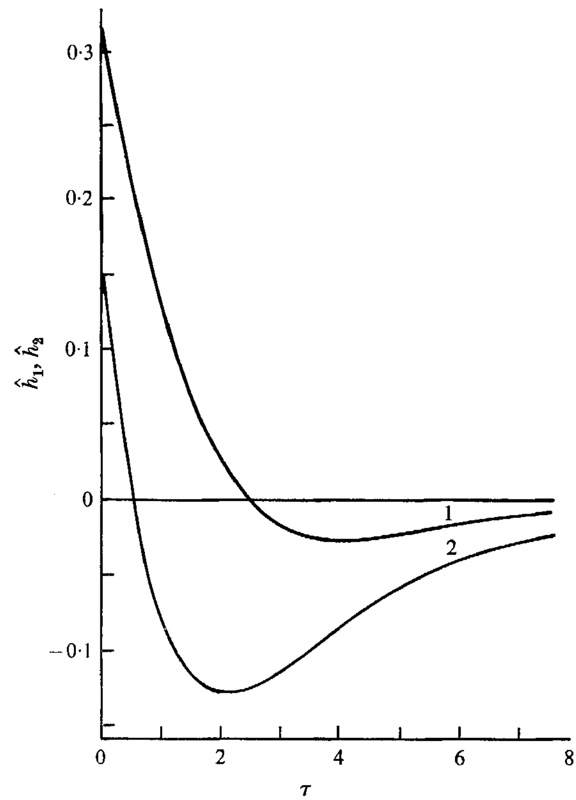

FIGURE 4. The integral components 1, $\hat{h}_1(\tau)$ and 2, $\hat{h}_2(\tau)$. The integral component $\hat{h}_j(\tau)$ ($j = 1, 2$) is the difference between $h_j(\tau)$ and the polar component of $h_j(\tau)$.

of $h_1(\tau)$ is dominant when $(8/\pi)\tau^{-3} > 0{\cdot}8818 \exp(-0{\cdot}1309\tau)$, i.e. when $\tau > 97$, and similarly the integral component of $h_2(\tau)$ is dominant when $\tau > 61$, after approximately 9 cycles. The order of magnitude of $h_2(\tau)$ at $\tau = 61$ is 3×10^{-4} which is negligible; thus the final stage of decay is not of practical importance.

6. Discussion

These results can be used to assess the accuracy of certain approximations which have been used in ship hydrodynamics. In these it is supposed (see, e.g. Havelock 1942) that the motion is governed by a second-order differential equation with constant coefficients; in other words, that the motion is damped harmonic. We now see that the best approximation of this kind is obtained by retaining the polar component and neglecting the integral component. The discrepancy which can be seen from figures 2 and 3, and from figure 4, can be reduced by crudely combining the polar component with the initial approximation during the first quarter-cycle. When this is done the maximum error in the approximation to $h_1(\tau)$ is reduced to 0·03 (attained near the first maximum) but the maximum error in $h_2(\tau)$ only to 0·1 (attained near the first minimum). This will be too crude for some applications.

This discussion presupposes, moreover, that the location of the pole and the values of the residues at the pole are known exactly from computations in the complex u-plane. In practice the pole, i.e. the zero of $F(u) = 1-\frac{1}{4}\pi u^2(1+\Lambda(u))$ may be roughly approximated in the following manner. The values of $\operatorname{Re}\Lambda(u)$ and $\operatorname{Im}\Lambda(u)$ on the real u-axis are supposed known from published values of the virtual-mass and wave-making coefficients, e.g. Ursell (1957). The first approximation u_1 is taken near that real value of u where $1-\frac{1}{4}\pi u^2(1+\operatorname{Re}\Lambda(u))$ vanishes, and the tangent approximation

$$u_2 = u_1 - [F(u_1)/F'(u_1)]$$

is then found. Thus, taking $u_1 = 0{\cdot}88$ and using accurate values of $F(u_1)$ and $F'(u_1)$, we find that $u_2 = 0{\cdot}895-0{\cdot}134i$, which may be compared with the exact value $0{\cdot}9117-0{\cdot}1309i$, see appendix 2 below.

Our numerical work may also be compared with the work of Sretenskii (1937) described by Wehausen & Laitone (1960, pp. 619–620) and also more briefly in I, § 1. Sretenskii obtained a pair of linear integro-differential equations and then introduced a thin-body assumption which has been criticized by Wehausen & Laitone but which enabled him to reduce the problem to a single equation which was solved numerically. His curve is reproduced by Wehausen & Laitone who draw attention to the difference between a damped harmonic oscillation and the solution of Sretenskii's integro-differential equation. This difference is not borne out by our calculations which show that the motion is very nearly damped-harmonic except during an initial and a final stage. It should, however, be noted that Sretenskii's calculations refer to a thin-body approximation and that no direct comparison with our work is therefore possible.

There remains the problem of finding the motion of the fluid, which may now be considered as being due to the *known* motion of the body. We have not yet studied this problem. We made a brief attempt at a direct solution of the combined body-fluid problem in which the equations (2.1)–(2.4) were replaced by finite-difference approximations in (x, y, t) space but it was quickly seen that this problem is a very big one. Our method gives the body displacement accurately while giving little information about the fluid motion.

We are indebted to Mr Ian Gladwell for help and advice with the computations. We also wish to express our gratitude to the Science Research Council for the research studentship held by one of the authors (S. J. Maskell) during the period of this research.

Appendix. Notes on numerical computation

1. The computation of $\Lambda(u)$ followed the method described in I, appendix 1. The infinite system obtained there (see I, top of p. 316) was truncated and solved numerically. When $|Ka| = |u^2|$ was progressively increased it was found necessary to use progressively more equations. The method worked well for both real and complex values of u.

2. The pole (i.e. the zero of $F(u)$) was found by an adaptation of the rule of false position to complex functions. The starting values for the zero of $F(u)$ were taken to be $0{\cdot}87-0{\cdot}10i$ and $0{\cdot}90-0{\cdot}14i$. Five iterations gave $u = 0{\cdot}91166-0{\cdot}13090i = b$ say, for which $|F(u)| < 10^{-5}$. It was found that $F'(b) = -2{\cdot}0904+0{\cdot}7638i$.

3. Let the integral components be denoted by $\hat{h}_1(\tau)$ and $\hat{h}_2(\tau)$ respectively. Thus, for example,

$$\hat{h}_1(\tau) = \frac{1}{2\pi}\left\{\int_{-\infty}^{0} e^{-iut}f(u)\,du + \int_{0}^{\infty} e^{-iut}f(u)\,du\right\}, \tag{A 1}$$

where $f(u) = [1-\frac{1}{4}\pi u^2(1+\Lambda(u))]^{-1}$, and where the integrations are along $\arg u = \pi+\tan^{-1}\frac{1}{2}$ and $\arg u = -\tan^{-1}\frac{1}{2}$ respectively. For small values of the parameter τ and large values of u these integrals converge very slowly, like $u^{-2}e^{-iu\tau}$, and we were at first unable to achieve acceptable accuracy. This difficulty was overcome in the following manner for the integral (A 1) for $\hat{h}_1(\tau)$, and similarly for $\hat{h}_2(\tau)$.

The behaviour of $f(u) = [1-\frac{1}{4}\pi u^2(1+\Lambda(u)]^{-1}$ for large u can be found by using the asymptotic formula (3.8) which gives $f(u) \sim -(2/\pi u^2)-(16/3\pi^2u^4)$. We chose constants A_1 and B_1 such that

$$\frac{A_1}{1+u^2}+\frac{B_1}{(1+u^2)^2} \sim \frac{2}{\pi u^2}+\frac{16}{3\pi^2u^4},$$

i.e.

$$\frac{A_1}{u^2}\left(1-\frac{1}{u^2}\right)+\frac{B_1}{u^4} \sim \frac{2}{\pi u^2}+\frac{16}{3\pi^2u^4},$$

whence $A_1 = 2/\pi$, $B_1 = 2/\pi+16/3\pi^2$. We now write

$$\hat{h}_1(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-iu\tau}du}{1-\frac{1}{4}\pi u^2(1+\Lambda)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{1}{1-\frac{1}{4}\pi u^2(1+\Lambda)}+\frac{A_1}{1+u^2}+\frac{B_1}{(1+u^2)^2}\right\}e^{-iu\tau}du \tag{A 2}$$

$$-\frac{A_1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-iu\tau}}{1+u^2}du-\frac{B_1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-iu\tau}}{(1+u^2)^2}du. \tag{A 3}$$

The integrand in (A 2) now decreases faster than u^{-4}, probably like u^{-6}, and thus numerical integration is feasible. The definite integrals in (A 3) are known functions:

$$\int_{-\infty}^{\infty}\frac{e^{-iu\tau}}{1+u^2}du = \pi e^{-\tau}, \quad \int_{-\infty}^{\infty}\frac{e^{-iu\tau}}{(1+u^2)^2}du = \tfrac{1}{2}\pi(1+\tau)e^{-\tau},$$

as can be shown by contour integration in the lower half u-plane where $u = -i$ is the only pole. Similarly the integral for $\hat{h}_2(\tau)$ can be treated, by adding an expression of the form

$$\frac{A_2 u}{1+u^2}+\frac{B_2 u}{(1+u^2)^2}+\frac{C_2 u}{(1+u^2)^3}$$

to the integrand. Details are given in Maskell (1969).

Checks were applied near $\tau = 0$ and near $\tau = \infty$ to test the accuracy of the computed values of $\hat{h}_1(\tau)$. Near $\tau = 0$ the sum of $\hat{h}_1(\tau)$ and of the polar contribution (5.1) should agree with the power series (3.10). Near $\tau = \infty$ the value of $\hat{h}_1(\tau)$ should be close to the asymptotic value $-(8/\pi)\tau^{-3}$. The corresponding checks were also applied to $h_2(\tau)$. The agreement was satisfactory.

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